Journey towards algebraic geometry seminar

II: Varieties & sheaves

Lecture 6: Nonsingular varieties

Speaker: Ryan Braiden.

Structure:

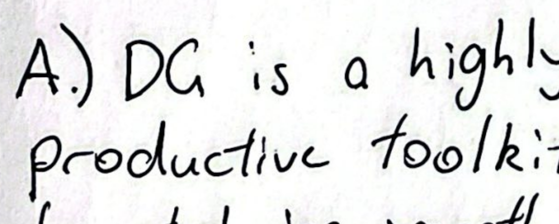
1. A brief tangent about tangents.
2. Nonsingular varieties.
3. How nonsingular is a variety?

---

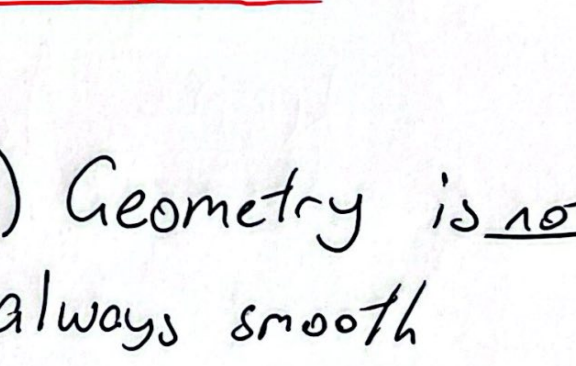
1 A brief tangent about tangents

Two truths:

A.) DG is a highly productive toolkit for studying smooth manifolds



B.) Geometry is not always smooth



Our goal: develop a theory of geometry that can study general geometric objects that may non-smooth or (even!) ~~non~~ arithmetical in description.

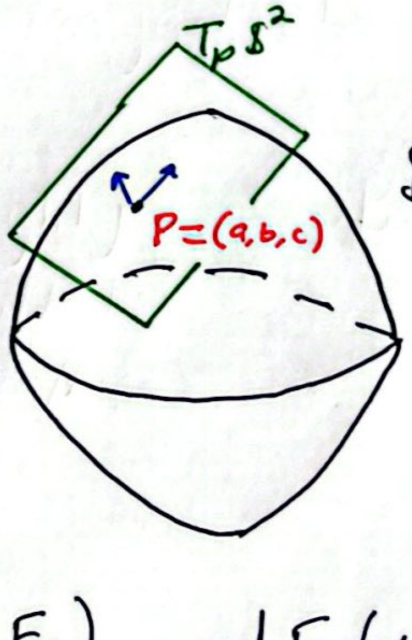
NEVERTHELESS, we want this theory to describe smoothness.

Question: How do we characterise smoothness on an affine variety?

Let's consider examples from DG.

1.) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(x,y,z) \longmapsto x^2+y^2+z^2-1$.

smooth



$$S^2 = F^{-1}(0)$$

$T_p S^2 = \ker(dF_p)$, $dF_p(u_1,u_2,u_3)$

$$= 2au_1 + 2bu_2 + 2cu_3.$$

③

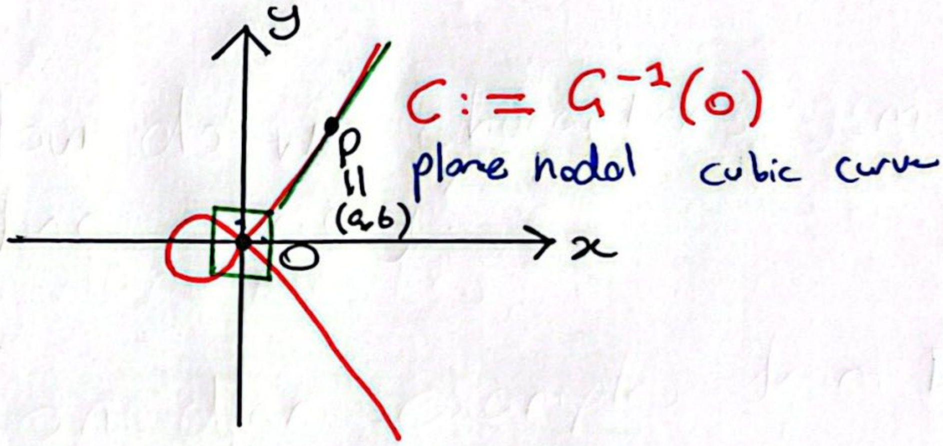
Since $\operatorname{rank}(dF_p) = 1$,

$$\dim T_p S^2 = \operatorname{null}(dF_p) = 3 - 1 = 2.$$

Notice:

$$\dim T_p S^2 = \dim S^2 \quad \forall\, p \in S^2.$$

2.) $G : \mathbb{R}^2 \longrightarrow \mathbb{R}, \ (x,y) \longmapsto y^2 + x^2(x+1)$



$T_pC = \ker(dG_p)$, $\quad dG_p(u_1, u_2)$

$$= 2bu_1 - (3a^2 + 2a)u_2.$$

If $P \neq O$, then $\operatorname{rank}(dG_p) = 1$.

If $P = O$, then $\operatorname{rank}(dG_p) = 0$.

$$\dim T_p C = \begin{cases} 1 & \text{if } P \neq O \\ 2 & \text{if } P = O \end{cases}.$$

At $P = O$, C does <u>NOT</u> resemble a 1-dim object.

These examples suggest, tangent spaces provide a tool for detecting smoothness of curves & surfaces in $\mathbb{R}^n$.

In moral language, smoothness is when

"local dimension = global dimension"

Since the Jacobian matrix at P $J(P)$ is the matrix rep of dF_P, we have the following criterion:

$$F^{-1}(0) \text{ smooth at } P \iff \overset{\ker}{\cancel{\text{rank}}}\, J(P) = \dim F^{-1}(0).$$

---

2 Nonsingular varieties

The above discussion provides a natural criterion for smoothness in the context of affine varieties.

Def'n (Nonsingular variety; extrinsic): $Y \subseteq \mathbb{A}^n$ affine variety, where $I(Y) = (f_1, \dots, f_t)$. Y is nonsingular at $P \in Y$ if

$$\dim Y = \ker \| (\partial f_i / \partial x_j)(P) \|, \text{ or equivalently,}$$

$$\operatorname{rank}\left(\left\| \left(\frac{\partial f_i}{\partial x_j}\right)(P) \right\| \right) = n - \dim Y.$$

Furthermore, Y is nonsingular if it is nonsingular at every point.

Some initial remarks

⊛ Partial derivatives are purely algebraic

$$\frac{\partial}{\partial x_i}\left(x_i^{\,n}\right) = n x_i^{\,n-1}.$$

Caution: if k has characteristic p, then

$$\frac{\partial}{\partial x_i}\left(x_i^{\,p}\right) = p x_i^{\,p-1} = 0.$$

(Rate of change interpretation of $\frac{\partial}{\partial x_i}$ is NOT valid).

⊛

$\cancel{J(P) = \|(\ldots)}$

$$\operatorname{rank} J(P) = \operatorname{rank} \left\| \left(\frac{\partial f_i}{\partial x_j}\right)(P) \right\|$$

is independent of generators of $I(Y)$.

⊛ This ~~defn~~ criterion depends on how Y is embedded in affine space.

↳ To determine if Y is nonsingular, we must know smth about its ambient space.

↳ An extrinsic notion of smoothness.

Example. ~~$f(x,y,z)$~~

$f : \mathbb{A}^2_{\mathbb{C}} \longrightarrow \mathbb{C}$, $f(x,y) = y^2 - x^2(x+1)$.

Is $X := Z(f) \subseteq \mathbb{A}^2_{\mathbb{C}}$ nonsingular?

f is irreducible $\Longrightarrow \dim X = 2 - 1 = 1$.

If $P = (a,b) \in X$, X is nonsingular at P iff

$$\operatorname{rank}\left(-a(3a+2),\ 2b\right) = 1.$$

If $P = 0$, $J(P) = (0,0)$, so $\operatorname{rank}(J(P)) = 0$.
X is singular at $P = 0$.

If ~~$P \neq 0$, &~~ $b \neq 0$, then

$2b \neq 0$ is a 1×1-minor (non-zero)

$\Longrightarrow \operatorname{rank}(J(P)) = 1$. ✓

If, $b = 0$ & $a \neq 0$, the 1×1-minor

$$-a(3a+2) = 0 \iff a = -\frac{2}{3}$$

but $(-\frac{2}{3}, 0) \notin X$. So $\operatorname{rank}(J(P)) = 1$ ✓.

$$\operatorname{Sing}(X) = \{(0,0)\}.$$

Question: How do we extend the notion of nonsingularity to any variety?

⊛ We need an <u>intrinsic</u> notion.

Theorem: $Y \subseteq \mathbb{A}^n$ affine variety, $P \in Y$.

Y is nonsingular at P $\iff$ $\mathcal{O}_{P,Y}$ is a regular local ring $\left(\dim_k \frac{m_P}{m_P^2} = \dim \mathcal{O}_{P,Y}\right)$

Quick aside: m_P / m_P^2 is a k-vector space which should be thought of as the cotangent space of Y at P.

"Y is nonsingular at P iff

$$\dim \text{"} T_P^* Y \text{"} = \dim Y \text{."}$$

Proof of Theorem

Need to handle

m_P / m_P^2 as a k-vector space.

Let $\mathfrak{a}_p := I(p)$ & $\mathfrak{b} := I(Y)$. Recall:

$$\circledast \quad \mathcal{O}_{p,Y} \simeq \left(A/\mathfrak{b}\right)_{\mathfrak{a}_p/\mathfrak{b}} \underset{\text{w/ quotients}}{\overset{\text{(localization commutes)}}{\simeq}} \frac{A_{\mathfrak{a}_p}}{\mathfrak{b}A_{\mathfrak{a}_p}}$$

$$\tilde{\circledast} \quad \dim \mathcal{O}_{p,Y} = \dim Y.$$

From $\circledast$ it follows that

$$\underbrace{\frac{m_p}{m_p^2} \simeq_{\mathrm{Vect}_k} \frac{\mathfrak{a}_p}{(\mathfrak{b}+\mathfrak{a}_p^2)}}_{\text{Exercise in commutative algebra.}}$$

Hence

$$\dim_k \frac{m_p}{m_p^2} = \dim_k \mathfrak{a}_p - \dim_k(\mathfrak{b}+\mathfrak{a}_p^2) + \dim_k(\mathfrak{a}_p^2) - \dim_k(\mathfrak{a}_p^2)$$

$$= \dim_k\left(\frac{\mathfrak{a}_p}{\mathfrak{a}_p^2}\right) - \dim_k\left(\frac{\mathfrak{b}+\mathfrak{a}_p^2}{\mathfrak{a}_p^2}\right).$$

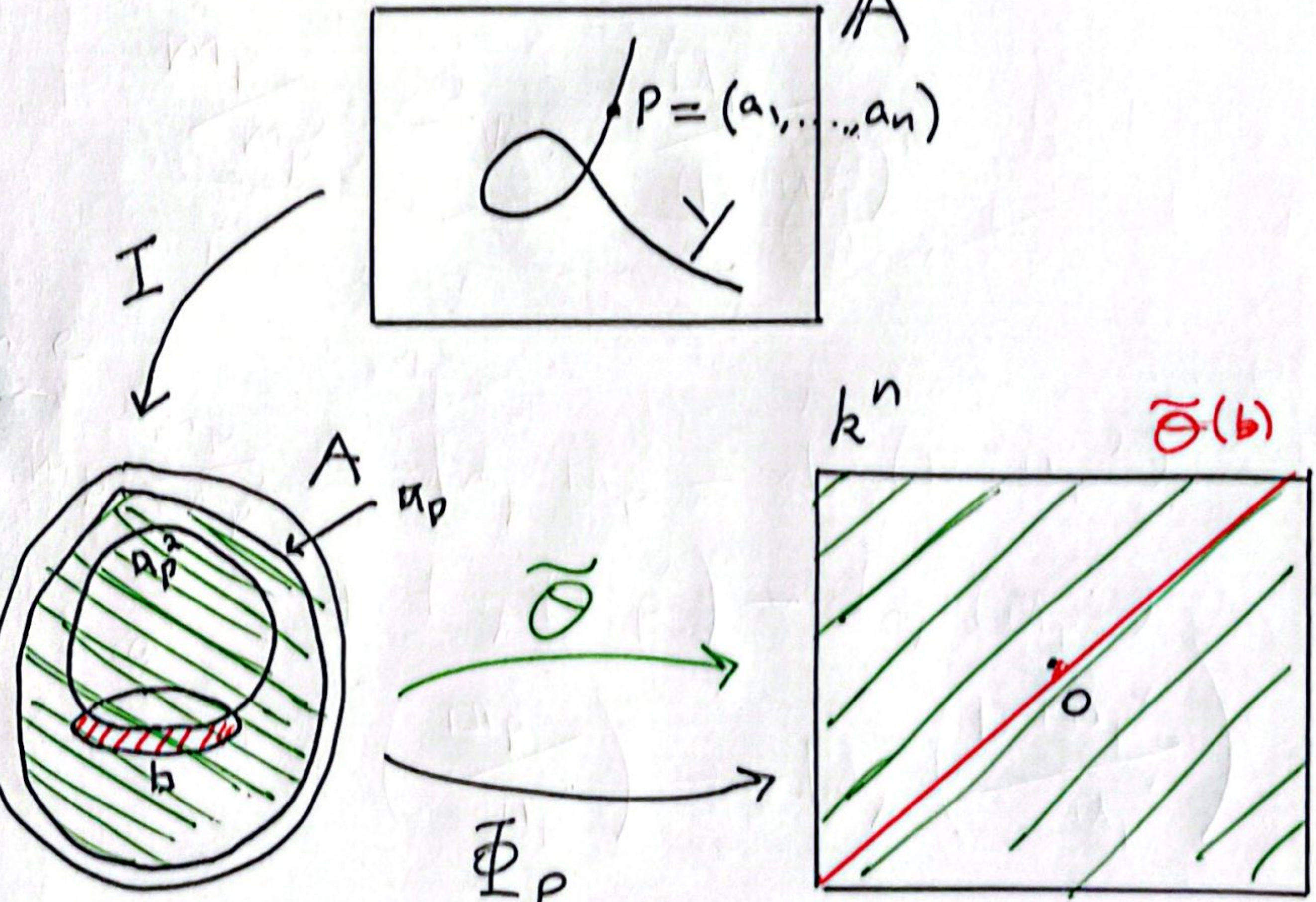
It suffices to study the spaces

$$\frac{\mathfrak{a}_p}{\mathfrak{a}_p^2} \quad \& \quad \frac{\mathfrak{b}+\mathfrak{a}_p^2}{\mathfrak{a}_p^2}.$$

In addition, we need to relate these spaces to the Jacobian matrix of Y. There is a linear differential map

$$\Phi_p : A \longrightarrow k^n, \quad f \longmapsto \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Let's visualise this:



Note that

$$\{\Phi_p(x_i - a_i) : 1 \leq i \leq n\}$$

spans k^n. Hence $\tilde{\theta} := \Phi_p|_{\mathfrak{m}_p}$ is a surjective linear map.

Question: Is $\tilde{\theta}$ an isomorphism? No.

$$\tilde{\theta}(x_i) = \tilde{\theta}(x_i + a) \quad \forall\, a \in k.$$

What is its kernel? If $f \in m_p$, then

$$f = \sum_j g_j (x_j - a_j), \quad g_j \in A.$$

If $\tilde{\theta}(f) = 0$

$$\Rightarrow \sum_j \frac{\partial (g_j (x_j - a_j))}{\partial x_i}(P) = 0 \quad \forall\, 1 \le i \le n.$$

Note

$$\sum_j \frac{\partial (g_j (x_j - a_j))}{\partial x_i}(P) = \sum_j \Big(\frac{\partial g_j}{\partial x_i}(P) \underbrace{(a_j - a_j)}_{0} + g_j(P) \frac{\partial (x_j - a_j)}{\partial x_i}(P) \Big)$$

$$= g_i (P).$$

So $g_i \in m_p$, $\forall\, 1 \le i \le n$. Hence

$$\ker \tilde{\theta} = m_p^2.$$

We have

$$\theta : \frac{m_p}{m_p^2} \xrightarrow{\sim} k^n, \quad \theta(f + m_p^2) = \tilde{\theta}(f).$$

⊛ Update:

$$\dim_k\left(\frac{\mathfrak{a}_P}{\mathfrak{a}_P^2}\right) = \dim_k(k^n) = n$$

$$\Longrightarrow \dim_k\left(\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}\right) = n - \dim\left(\frac{\mathfrak{b}+\mathfrak{a}_P^2}{\mathfrak{a}_P^2}\right).$$

Write $\mathfrak{b} = (f_1, \dots, f_t)$. Then $\tilde{\theta}\big|_{\mathfrak{b}}$ is the matrix rep of $J(P)$. Hence

$$\dim_k \tilde{\theta}(\mathfrak{b}) = \operatorname{rank} J(P).$$

We claim

$$\underbrace{\theta^{-1}\left(\tilde{\theta}(\mathfrak{b})\right) \simeq \frac{\mathfrak{b}+\mathfrak{a}_P^2}{\mathfrak{a}_P^2}}_{\text{Straightforward exercise in commutative algebra.}}$$

i.e.,

$$\dim_k\left(\frac{\mathfrak{b}+\mathfrak{a}_P^2}{\mathfrak{a}_P^2}\right) = \dim_k \tilde{\theta}(\mathfrak{b}) = \operatorname{rank} J(P).$$

$$\Longrightarrow \dim_k\left(\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}\right) = n - \operatorname{rank} J(P).$$

Thus

Y is non-singular at P

$\Longleftrightarrow \dim_k\left(\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}\right) = n-(n-r) = r = \dim Y = \dim \mathcal{O}_{P,Y}$

$\Longleftrightarrow \mathcal{O}_{P,Y}$ is a regular local ring. $\square$

Def^n (Nonsingular; intrinsic): Let Y be any variety. Y is nonsingular at $P \in Y$ if $\mathcal{O}_{P,Y}$ is a regular local ring. Y is nonsingular if it is nonsingular at every point. Y is singular if it is not nonsingular.

Example:

Consider the projective variety

$$X = Z(\overbrace{y^2z - x^3 - x^2z}^{\text{irreducible}}) \subseteq \mathbb{P}^2_{\mathbb{C}}$$

So

$$\dim \mathcal{O}_{P,X} = \dim X = 2 - 1 = 1.$$

We need to compute the $\mathbb{C}$-vector space

$$\frac{m_P}{m_P^2},$$ where m_P is the maximal ideal of $\mathcal{O}_{P,X}$.

~~It suffices to compute~~

Observation: Given the affine chart

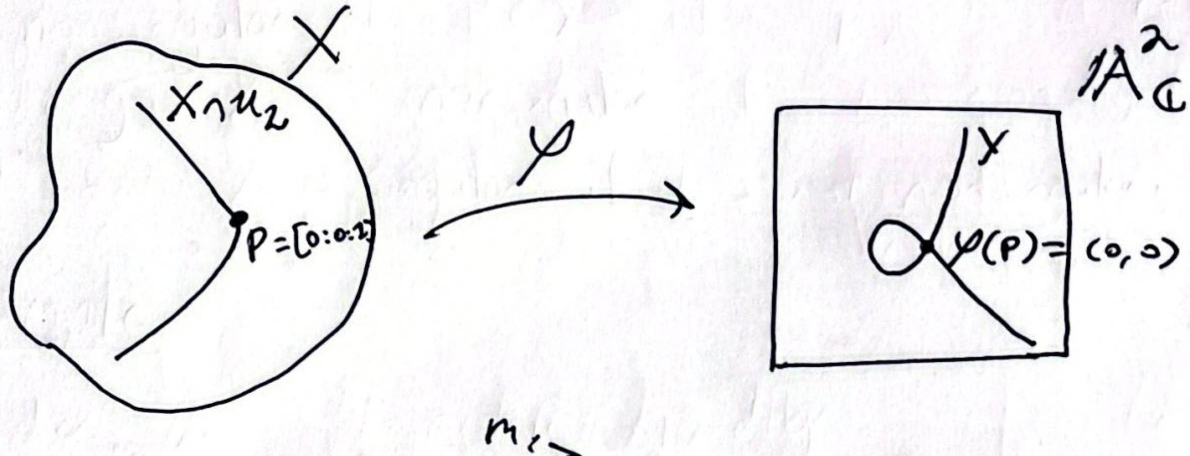
$$\mathcal{U}_z := \mathbb{P}^2_{\mathbb{C}} - Z(z),$$

there is an iso. of varieties

$$\varphi : X \cap \mathcal{U}_z \xrightarrow{\sim} Y := Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$$

$$[x : y : z] \longmapsto \left(\frac{x}{z}, \frac{y}{z}\right).$$

Noting that $\varphi([0:0:1]) = (0,0)$



Since $I(\varphi(P)) = \overset{m :=}{(x,y)}$, we have

$$\mathcal{O}_{P,X} \simeq A(Y)_{(x,y)}.$$

It is sufficient to compute the $\mathbb{C}$-vector space

$$\frac{m}{m^2} = \frac{(x,y)}{(x^2, xy, y^2)}$$

Since

$$\dim_{\mathbb{C}} \frac{m_P}{m_P^2} = \dim_{\mathbb{C}} \frac{m}{m^2}.$$ (Exercise in commutative algebra)

Any $f \in m = (x,y)$, $\exists\ a, b \in \mathbb{C}$ s.t.

$$f = ax + by + g$$

where $g \in m^2$. The map

$$\phi : m \longrightarrow \mathbb{C}^2,\ f \longmapsto (a,b)$$

is linear, surjective & $\ker \phi = m^2$. So

$$\frac{m}{m^2} \simeq \mathbb{C}^2.$$

Hence,

$$\dim_{\mathbb{C}} \frac{m_p}{m_p^2} = \dim_{\mathbb{C}} \mathbb{C}^2 = 2 \neq \dim \mathcal{O}_{p,X} = 1$$

where $P = [0:0:1]$. Hence X is nonsingular at $[0:0:1]$.

---

③ How nonsingular is a variety?

Theorem:

$\mathrm{Sing}(Y)$ is a proper closed subset of Y.